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On averaged equations for finite-amplitude water waves

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The wave-action conservation equation for water waves is always derived from a Lagrangian for irrotational flow. This is quite satisfactory if the whole flow-field (i.e. waves and background current) is irrotational, but is inadequate for a background current with a large-scale (vertical) vorticity, even if the flow has negligible vorticity on the local scale of a few wavelengths. A wave-action conservation equation is derived for this case and equations governing the flow and the waves are given in a simple form closely parallel to the irrotational flow equations.

1. Introduction

The behaviour of very slowly-varying water-wave trains is usually studied by using one of two available sets of averaged equations. Either Phillips' (1966, § 3.6) directly averaged equations of motion, or Whitham's (1974, § 16.7) equations derived from an averaged Lagrangian. Whitham's set, thanks to its elegance, seems to be more attractive than Phillips' more general but cumbersome set. One should bear in mind that the Lagrangian used by Whitham is only for irrotational flow. Both sets are conveniently presented in a recent paper by Crapper (1979) who extends them to include surface tension and shows that Whitham's equations can be manipulated into the same equations as Phillips' set.

In this paper the two sets of equations are compared in § 2. This is followed in § 3 by the equations closely resembling, but not identical to, Whitham's that can be obtained from Phillips' equations. The difference is that Whitham's consistency equations for the current, which include the irrotationality condition, are modified. All other equations are unchanged. Finally, in § 4, versions of the equations are given for steady problems with variation in one direction only.

2. Comparison between Phillips' and Whitham's sets of equations

In considering the general problem of waves propagating over water of slowly-varying depth h with pre-existing currents, the following seven unknowns are usually chosen as dependent variables: a wave-amplitude measure a, the wave frequency ω, wavenumber components (k_1, k_2), the average water depth D, and current velocity components (U_1, U_2). These last three cannot be specified *ab initio* since finite-amplitude waves generate and modify currents.

To determine these unknowns requires seven equations. Three are of a kinematic

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nature, and common to both approaches. They arise as consistency conditions from the definition of k_1, k_2 and ω as derivatives of a phase function and are

$$\frac{\partial k_1}{\partial x_2} = \frac{\partial k_2}{\partial x_1} \tag{1}$$

and

$$\frac{\partial k_\alpha}{\partial t} + \frac{\partial \omega}{\partial x_\alpha} = 0 \quad \alpha = 1, 2, \tag{2}$$

where ω is defined by the Doppler relation,

$$\omega = \sigma + U_\alpha k_\alpha,$$

and $\sigma = \sigma(a, k, D)$ is the dispersion relation in the absence of current, where $k = |\mathbf{k}|$. There is an important non-uniqueness here, since one is at liberty to define what is meant by zero current in water of finite constant depth. We follow all the authors cited here in using the definition that the mean horizontal velocity at a fixed point below the troughs of the waves is equal to the current, U_α. Thus U_α is the Eulerian-mean current. An alternative definition is to take the total mass transport divided by ρD to be the current, i.e. zero current = zero mass transport: this gives a different result and is not used here.

The additional four equations in Phillips' approach, obtained by averaging the equations of motion over depth and phase can be written in the following forms.

Conservation of total mass:

$$\rho \frac{\partial D}{\partial t} + \frac{\partial}{\partial x_\beta}(\rho D U_\beta + I_\beta) = 0. \tag{3}$$

Conservation of total momentum:

$$\frac{\partial}{\partial t}(\rho D U_\alpha + I_\alpha) + \frac{\partial}{\partial x_\beta}\left[(\rho D U_\alpha + I_\alpha)\left(\frac{I_\beta}{\rho D} + U_\beta\right) + \tfrac{1}{2}\rho g D^2 \delta_{\alpha\beta} + S_{\alpha\beta} - \frac{I_\alpha I_\beta}{\rho D}\right] = \rho g D \frac{\partial h}{\partial x_\alpha}. \tag{4}$$

Conservation of total energy:

$$\frac{\partial}{\partial t}[\tfrac{1}{2}\rho D U^2 + \tfrac{1}{2}\rho g (D-h)^2 + T + V + U_\alpha I_\alpha] + \frac{\partial}{\partial x_\alpha}\{U_\alpha[\tfrac{1}{2}\rho D U^2 + \rho g D(D-h) + T + V + U_\beta I_\beta] + F_\alpha + I_\alpha[g(D-h) + \tfrac{1}{2}U^2] + S_{\alpha\beta} U_\beta\} = 0. \tag{5}$$

The averaged wave quantities, momentum I_α, kinetic and potential energies T, V, and radiation stress or wave-induced momentum flux $S_{\alpha\beta}$ are defined in appendix A, and $y = -h(x_\alpha)$ is the bottom topography. The equations may be readily derived from Phillips' (1966) equations (3.6.4, 11 and 18) or Crapper's (1979) equations (5, 16 and 27). There are many ways of writing the above equations. The above arrangement of terms is such that the left-hand side of each is in conservation form, and within each expression the terms are in the order: current, wave, and interaction terms.

If Whitham is followed, the velocity U_α and a quantity γ appear in the averaged Lagrangian,

$$\gamma = g(D-h) + \tfrac{1}{2}U^2 + \tfrac{1}{2}\overline{u_b^2} \tag{6}$$

where u_b is the water velocity due to the waves at the bottom. In all presentations of this approach, e.g. Crapper (1979, § 3) or Whitham (1974, § 16.7) relations equivalent to consistency relations for a pseudo-phase are derived. These are

$$\frac{\partial U_\alpha}{\partial t}+\frac{\partial \gamma}{\partial x_\alpha}=0 \tag{7}$$

and

$$\frac{\partial U_1}{\partial x_2}=\frac{\partial U_2}{\partial x_1}. \tag{8}$$

Equation (8) implies a global restriction to irrotational flow whereas the direct approach from the equations of motion only involves assuming that the wave-motion, on a local scale, is irrotational.

Whitham's formulation gives the mass conservation equation (3); in place of momentum conservation there are equations (7) and in place of the energy equation there is wave-action conservation, which may be written

$$\frac{\partial}{\partial t}\left(\frac{I}{k}\right)+\frac{\partial}{\partial x_\alpha}\left[U_\alpha\frac{I}{k}+(3T-2V+\tfrac{1}{2}\rho D\overline{u_b^2})\frac{k_\alpha}{k^2}\right]=0. \tag{9}$$

It is interesting to note that if a factor of 2π is introduced then wave-action, $2\pi I/k$, is the momentum per wave relative to the mean flow U_α.

Crapper (1979) shows that Whitham's equations are consistent with Phillips'. Here we show that for a flow with large-scale vorticity the wave-action equation (9) still holds and deduce the new equations needed corresponding to equations (7) and (8).

3. Derivation of equations

It proves necessary to use all the exact relations between the integral properties of water waves that are derived by Longuet-Higgins (1975). For convenient reference some are given in appendix B. The algebra is long and tedious; only an outline is given here.

The initial part of the algebra, to transform the energy equation into a form close to that of the wave-action equation, may be summarized as

$$\frac{1}{\omega}\left\{(E5)-\gamma(E3)-\left[\frac{IU_\alpha}{k}+(3T-2V+\tfrac{1}{2}\rho D\overline{u_b^2})\frac{k_\alpha}{k^2}\right](E2)_\alpha\right\},$$

where, for example, $\gamma(E3)$ stands for γ times equation (3), γ still being defined by (6). Using $2T=cI$ and the three other wave relations given in appendix *B* eventually leads to

$$\left\{\frac{\partial}{\partial t}\left(\frac{I}{k}\right)+\frac{\partial}{\partial x_\alpha}\left[\frac{I}{k}U_\alpha+(3T-2V+\tfrac{1}{2}\rho D\overline{u_b^2})\frac{k_\alpha}{k^2}\right]\right\}+\frac{1}{\omega}(\rho DU_\alpha+I_\alpha)\left(\frac{\partial\gamma}{\partial x_\alpha}+\frac{\partial U_\alpha}{\partial t}\right)=0. \tag{10}$$

However, from the momentum equation, more precisely from $\{(E4)-U_\alpha(E3)_\alpha\}/\rho D$, after using equations (2), (B 2) and (B 3), we obtain

$$\frac{k_\alpha}{\rho D}\left\{\frac{\partial}{\partial t}\left(\frac{I}{k}\right)+\frac{\partial}{\partial x_\beta}\left[\frac{I}{k}U_\beta+(3T-2V+\tfrac{1}{2}\rho D\overline{u_b^2})\frac{k_\beta}{k^2}\right]\right\}$$
$$+\frac{\partial U_\alpha}{\partial t}+\frac{\partial\gamma}{\partial x_\alpha}+\left(\frac{\partial U_\alpha}{\partial x_\beta}-\frac{\partial U_\beta}{\partial x_\alpha}\right)\left(U_\beta+\frac{I_\beta}{\rho D}\right)=0. \tag{11}$$

In equations (10) and (11) the term in curly brackets is just the left-hand side of equation (9) and is eliminated from these equations, in

$$\left(U_\alpha+\frac{I_\alpha}{\rho D}\right)\left\{(E11)_\alpha-\frac{k_\alpha}{\rho D}(E10)\right\}$$

to give

$$(\rho D U_\alpha + I_\alpha)\left(\frac{\partial \gamma}{\partial x_\alpha}+\frac{\partial U_\alpha}{\partial t}\right) = 0. \tag{12}$$

So equation (10) becomes identical to equation (9). Thus, wave-action, defined as I/k, is conserved for waves on a rotational current.

Having proved equation (9) to be valid we see that equations (11) yield the following equations,

$$\frac{\partial U_\alpha}{\partial t}+\frac{\partial \gamma}{\partial x_\alpha}+\left(\frac{\partial U_\alpha}{\partial x_\beta}-\frac{\partial U_\beta}{\partial x_\alpha}\right)\left(U_\beta+\frac{I_\beta}{\rho D}\right) = 0. \tag{13}$$

Thus equations (12) and (13) replace the consistency equations (7) and (8) in order to make Whitham's approach fully equivalent to Phillips!

This last pair of equations can be rewritten to show the influence of the waves on the currents more directly:

$$\frac{\partial U_\alpha}{\partial t}+U_\beta\frac{\partial U_\alpha}{\partial x_\beta}+g\frac{\partial}{\partial x_\alpha}(D-h)+\left(\frac{\partial U_\alpha}{\partial x_\beta}-\frac{\partial U_\beta}{\partial x_\alpha}\right)\frac{I_\beta}{\rho D}+\frac{1}{2}\frac{\partial}{\partial x_\alpha}\overline{u_b^2} = 0.$$

The last two terms are zero if there are no waves. The last three terms become negligible for infinite depth and the current is uncoupled from the waves, as is assumed by Peregrine & Thomas (1979). However, the shallow-water approximation which is being used for the current is inappropriate as $h\to\infty$ and further analysis is required. This should be similar to the analysis involved in studies of short waves on longer waves, for example, see the discussion in Peregrine (1976, § II. F).

4. Discussion and simple examples

After the recent work on averaged equations and wave-action by Andrews & McIntyre (1978*a*, *b*) it is not surprising that conservation of wave-action is confirmed here for water waves on a rotational flow since they derive such a conservation relation for very general types of wave motion. The averaging operator used by Andrews & McIntyre to prove most of their results differs from that used in this paper and hence a direct use of their results is not possible in this context. Andrews & McIntyre's theory is applicable to slowly-varying water waves if averaging is confined to averaging with respect to phase and a dependence on the vertical coordinate retained. This may be very useful for studying the vertical distribution of wave-induced currents. The generalized Lagrangian-mean description introduced by Andrews & McIntyre suggests that it may also be valuable to consider water-wave motion relative to a reference frame in which there is zero mass flow associated with the waves. Jonsson (1978) also draws attention to this reference frame. The details of such a change are being investigated, but are not considered appropriate for this paper which aims to unify past work in this subject.

The new equations (12) and (13) to replace the 'pseudo-consistency' equations (7) and (8) are simple in appearance. It is possible to rearrange them into other simple combinations, but which particular equations are most suitable for use in any particular problem is likely to vary considerably.

Among the simplest examples of significant problems are those for steady wave fields, i.e. $\partial/\partial t = 0$, with variation in one direction only, i.e. $\partial/\partial x_2 = 0$. Equations (1) and (2) and the Doppler relation reduce to

$$k_2 = \text{constant}, \tag{14}$$

and

$$\omega = \sigma + U_\alpha k_\alpha = \text{constant}. \tag{15}$$

Mass conservation gives constant mass flux in the x_1 direction,

$$\rho D U_1 + I_1 = \text{constant} = Q_1, \text{ say}, \tag{16}$$

and similarly wave-action flux in the x_1 direction is

$$\frac{I}{k} U_1 + (3T - 2V + \tfrac{1}{2}\rho D\overline{u_b^2}) \frac{k_1}{k^2} = \text{constant}. \tag{17}$$

Equations (12) and (13) become

$$Q_1 \frac{d\gamma}{dx_1} = 0, \tag{18}$$

$$\frac{d\gamma}{dx_1} - \frac{dU_2}{dx_1}\left(U_2 + \frac{I_2}{\rho D}\right) = 0 \tag{19}$$

and

$$\frac{dU_2}{dx_1} Q_1 = 0. \tag{20}$$

These equations can be integrated in certain special cases. If the flow is irrotational, $dU_2/dx_1 = 0$, and equations (18) to (20) are satisfied by γ = constant. Irrotational flow includes many important examples such as waves approaching a beach, at any angle.

If there is vorticity, $dU_2/dx_1 \neq 0$, then $Q_1 = 0$. This constraint is independent of the wave field and is due to our implicit requirement that the velocity field satisfy the shallow-water equations in the absence of waves. Equation (19) is then the only non-trivial equation of (18) to (20). It is not directly integrable; however, equation (4) for $\alpha = 2$ is integrable, and gives

$$U_1 I_2 + S_{12} = \text{constant}, \tag{21}$$

and equation (4) for $\alpha = 1$ gives

$$\frac{d}{dx_1}\left(\tfrac{1}{2}\rho g D^2 + S_{11} - \frac{I_1^2}{\rho D}\right) = \rho g D \frac{dh}{dx_1}, \tag{22}$$

which may be more convenient than equation (19). This is particularly so if refraction of waves by currents alone is considered. That is, if $h(x_1)$ is constant, since then the right-hand side of (22) is zero and it may be integrated.

Appendix A. Definitions of integral wave properties

The following definitions are those of Longuet-Higgins (1975) and are also used by Cokelet (1977) and Crapper (1979). The mean wave momentum density is

$$I_\alpha = \overline{\rho \int_{-h}^{\eta} u_\alpha \, dy} \tag{A 1}$$

where (u_α, v) is the velocity field of the wave alone in the frame of reference where $\bar{u}_\alpha = 0$, $y = \eta(x_\alpha, t)$ is equation of the free surface and the overbar denotes an average over the phase $(k_\alpha x_\alpha - \sigma t)$ of the waves. The radiation stress tensor is

$$S_{\alpha\beta} = \overline{\int_{-h}^{\eta} (\rho u_\alpha u_\beta + p\delta_{\alpha\beta}) \, dy} - \tfrac{1}{2}\rho g D^2 \delta_{\alpha\beta} \tag{A 2}$$

where p is the pressure. The energy flux vector is

$$F_\alpha = \overline{\int_{-h}^{\eta} u_\alpha[\tfrac{1}{2}\rho(u_\beta u_\beta + v^2) + p + \rho g(y + h - D)] \, dz}. \tag{A 3}$$

The kinetic and potential energies are:

$$T = \overline{\int_{-h}^{\eta} \tfrac{1}{2}\rho(u_\alpha u_\alpha + v^2) \, dz}, \tag{A 4}$$

and

$$V = \tfrac{1}{2}\rho g[\overline{\eta^2} - (D - h)^2]. \tag{A 5}$$

Appendix B. Relations between the integral wave properties

$$F_\alpha = \frac{ck_\alpha}{k}(3T - 2V + \tfrac{1}{2}\rho D\overline{u_b^2}) + \tfrac{1}{2}I_\alpha \overline{u_b^2}, \tag{B 1}$$

$$S_{\alpha\beta} = \frac{k_\alpha k_\beta}{k^2}(3T - 2V + \tfrac{1}{2}\rho D\overline{u_b^2}) + \delta_{\alpha\beta}(T - V + \tfrac{1}{2}\rho D\overline{u_b^2}) \tag{B 2}$$

and

$$\partial(T - V) = I\partial c - k^{-1}(T - 2V + \tfrac{1}{2}\rho D\overline{u_b^2})\,\partial k - \tfrac{1}{2}\rho\overline{u_b^2}\,\partial D, \tag{B 3}$$

where $c = \sigma/k$ is the wave phase velocity. Equations (B 1) and (B 3) are given by Longuet-Higgins (1975) and equation (B 2) by Crapper (1979).

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